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Uniform pathwise connectedness and Whitney levels

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Abstract

It is proved that the property of being a uniformly pathwise connected continuum is a Whitney property, but is not a Whitney-reversible property. © 2001 Elsevier Science B.V. All rights reserved.

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Let X be a continuum, i.e., a compact connected metric space, and let $C(X)$ be the space of all nonempty subcontinua of X topologized by the Hausdorff metric.

A continuum X is said to be *uniformly pathwise connected* if there exists a family $\mathcal{F} = \{p : [a,b] \to X\}$ of paths in X, satisfying

$(*)$ for any $x, y \in X$ there is a path $p \in \mathcal{F}$ joining x with y, and

$(**)$ for any $\varepsilon > 0$ there is a positive integer n such that for each $p \in \mathcal{F}$ there are numbers $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ such that $\operatorname{diam} p([t_{i-1}, t_i]) \leqslant \varepsilon$ for $i = 1, 2, \ldots, n$.

It is an easy observation that the continuous images of the cone over the Cantor set are uniformly pathwise connected. The converse is also true and was proved by Kuperberg [8].

A map $\mu : C(X) \to [0, \infty)$ satisfying the conditions

$$\mu \textit{ is continuous on } C(X), \tag{1}$$

$$\textit{if } A \subset B \textit{ and } A \neq B, \textit{ then } \mu(A) < \mu(B), \tag{2}$$

$$\mu(\{x\}) = 0 \quad \textit{for every } x \in X, \tag{3}$$

is called a *Whitney map.*

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The Whitney maps are open and monotone (see Nadler [10, p. 400]). Then it is natural to study the following properties.

A property $\mathcal{P}$ is said to be a *Whitney property* provided that whenever a continuum X has property $\mathcal{P}$, so does $\mu^{-1}(t)$ for each Whitney map μ, for $C(X)$ and each t, $0 \leqslant t < \mu(X)$ (see [10, p. 399]).

A property $\mathcal{P}$ is said to be a *Whitney-reversible property* provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property $\mathcal{P}$ for all Whitney maps μ for $C(X)$ and all $0 < t < \mu(X)$, then X has property $\mathcal{P}$ (see [10, p. 453]).

It is known that arcwise connectedness is a Whitney property (see [10, p. 405]). In [5] Kato has proved that if X is uniformly pathwise connected continuum and has the property of Kelley (see [10, p. 538] for the definition), then for any Whitney map μ, and any $t \in [0, \mu(X))$, the level $\mu^{-1}(t)$ is uniformly pathwise connected. In this paper we prove that the same conclusion holds without the assumption that X has the property of Kelley. This answers in the positive the question posed by Illanes and Nadler in [4]. There are easy examples of uniformly pathwise connected continua without the property of Kelley.

It is known that arcwise connectedness is not a Whitney-reversible property (see [10, p. 457]). Illanes and Nadler asked in the book [4] if uniform pathwise connectedness is a Whitney-reversible property. In this paper we show that the assumption that all Whitney levels $\mu^{-1}(t)$, for $0 < t < \mu(X)$, are uniformly pathwise connected does not suffice for the level $\mu^{-1}(0)$ to have this property.

An *order arc* in $C(X)$ is an arc $\mathcal{A}$ in $C(X)$ such that for every pair of elements L, K of $\mathcal{A}$, we have $L \subset K$ or $K \subset L$. If A and B are subcontinua of X, and $A \subset B$, then there is an order arc in $C(X)$ beginning at A and ending at B (see [10, p. 63], where detailed references can be found).

By a path in X joining points $x, y \in X$ we mean a continuous function $p:[a,b] \to X$ such that $p(a) = x$ and $p(b) = y$.

The symbol $N_\delta(A)$ denotes the open δ-neighborhood of A in X; dist denotes the Hausdorff distance.

We will use the following lemma (cf. Krasinkiewicz [9, p. 157]).

Lemma 1. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if K and L are elements of $C(X)$ such that $K \subset N_\delta(L)$ and $|\mu(L) - \mu(K)| < \delta$, then* $\operatorname{dist}(K, L) < \varepsilon$.

Modifying the proof of (14.8.1) from [10, p. 405], we get the following.

Lemma 2. *Let μ be a fixed Whitney map and let $t \in (0, \mu(X))$. There exists a family $\mathcal{F}$ of paths $q:[0,t] \to \mu^{-1}(t)$ such that for any $\varepsilon > 0$ there exists a positive integer n such that if $A, B \in \mu^{-1}(t)$ and $A \cap B \neq \emptyset$ then we can choose a path $q \in \mathcal{F}$ such that $q(0) = B$ and $q(t) = A$ and such that the diameter of $q([(i-1)\frac{t}{n}, i\frac{t}{n}])$ is $\leqslant \varepsilon$, for $i = 1, \ldots, n$.*

Proof. If $A, B \in \mu^{-1}(t)$ and $A \cap B \neq \emptyset$ then for $x \in A \cap B$ consider order arcs $\mathcal{A}$ and $\mathcal{B}$ in $C(X)$ one with ends $\{x\}$ and A and other with ends $\{x\}$ and B. Let α and

β be parametrizations of these arcs by using the Whitney map, i.e., $\alpha:[0,t]\to\mathcal{A}$, and $\beta:[0,t]\to\mathcal{B}$, such that

$$\mu\bigl(\alpha(u)\bigr)=\mu\bigl(\beta(u)\bigr)=u, \tag{4}$$

for each $u\in[0,t]$. We have $\alpha(0)=\beta(0)=\{x\}$, and $\alpha(t)=A$ and $\beta(t)=B$.

For a fixed $u\in[0,t]$ the subset $\{\alpha(u)\cup\beta(w)\colon\ w\in[0,t]\}$, of $C(X)$ is connected. Thus, from the continuity of μ it easily follows that for any $u\in[0,t]$ there exists $s\in[0,t]$ such that

$$\mu\bigl(\alpha(u)\cup\beta(s)\bigr)=t.$$

Observe that the set $T(u)=\{s\colon\ \mu(\alpha(u)\cup\beta(s))=t\}$ is a closed subset of $[0,t]$. Thus, $s(u)=\sup T(u)$ belongs to $T(u)$ and therefore we can define a function $q:[0,t]\to\mu^{-1}(t)$, setting

$$q(u)=\alpha(u)\cup\beta\bigl(s(u)\bigr),\quad \text{for } u\in[0,t]. \tag{5}$$

Although $s(u)$ need not be continuous, the function q is continuous.

Indeed, let $u_0\in[0,t]$ and let u_n, where $n=1,2,\ldots$, be a sequence of points of $[0,t]$ convergent to u_0. Let us denote by v arbitrarily chosen point of condensation of the set $\{s(u_n)\colon\ n=1,2,\ldots\}$. The sequence $s(u_n)$ has a subsequence $s(u_{n_k})$ convergent to v. Thus, the sequence $q(u_{n_k})$ converges to $\alpha(u_0)\cup\beta(v)$, and $\alpha(u_0)\cup\beta(v)\in\mu^{-1}(t)$, for compactness of the level $\mu^{-1}(t)$.

Thus, to prove the continuity of q, it suffices to show that

$$\alpha(u_0)\cup\beta(v)=\alpha(u_0)\cup\beta\bigl(s(u_0)\bigr), \tag{6}$$

as v has been taken arbitrarily.

To do this, observe that $v\leqslant s(u_0)$, for $v\in T(u_0)$. Thus, $\alpha(u_0)\cup\beta(v)\subset\alpha(u_0)\cup\beta(s(u_0))$. Since the sets on both sides of this inclusion lie on $\mu^{-1}(t)$, hence, using (2) from the definition of the Whitney map, we obtain the equality (6).

Now, fix $\varepsilon>0$ and take $\delta>0$ according to Lemma 1. Then, for this δ take $\delta'>0$ according to Lemma 1. We have $\operatorname{dist}(K,L)<\delta$ for $K,L\in C(X)$ whenever $K\subset N_{\delta'}(L)$ and $|\mu(K)-\mu(L)|<\delta'$.

Let the integer n be such that $\frac{t}{n}<\delta'$.

To show that the number n is that one which we need for the function q, let u and w, where $u<w$, be points of $[0,t]$ such that

$$u \text{ and } w \text{ belong to } \Bigl[(i-1)\tfrac{t}{n}, i\tfrac{t}{n}\Bigr],\quad \text{for some } i\in\{1,\ldots,n\}. \tag{7}$$

Since $u<w$, then

$$\alpha(u)\subset\alpha(w) \tag{8}$$

and, by the monotonicity of the function s, we have $\beta(s(w))\subset\beta(s(u))$.

Since $w-u<\delta'$ (see (7)), then, by (4), $\mu(\alpha(w))-\mu(\alpha(u))<\delta'$. Thus, by (8) and Lemma 1, we have $\operatorname{dist}(\alpha(u),\alpha(w))<\delta$.

Therefore, using inclusions

$$\alpha(w)\subset N_\delta\bigl(\alpha(u)\bigr)\quad\text{and}\quad \beta\bigl(s(w)\bigr)\subset\beta\bigl(s(u)\bigr),$$

we obtain

$$\alpha(w) \cup \beta\big(s(w)\big) \subset N_\delta\big(\alpha(u) \cup \beta(s(u))\big),$$

which means that $q(w) \subset N_\delta(q(u))$.

Thus, having in view that $\mu(q(u)) = \mu(q(w))$, we can apply Lemma 1, and we obtain the required inequality $\operatorname{dist}(q(u), q(w)) < \varepsilon$. $\square$

Notice that n is universal for all points $A, B \in \mu^{-1}(t)$ such that $A \cap B \neq \emptyset$.

We say that a continuum Y is *uniformly continuum-chainable* if for each positive number ε there is an integer $k = k(\varepsilon)$ such that for each pair x, y of points of Y, there are subcontinua $A_1, \ldots, A_k$ of Y each of diameter less than ε and such that $x \in A_1$, $y \in A_k$ and $A_i \cap A_j \neq \emptyset$ whenever $|i - j| \leqslant 1$.

Theorem 1. *If a continuum X is uniformly continuum-chainable, then for any Whitney map μ and any $t \in (0, \mu(X))$, the Whitney level $\mu^{-1}(t)$ is uniformly pathwise connected.*

Proof. Assume that X is a uniformly continuum-chainable continuum. Let μ be a Whitney map for $C(X)$, and let $t \in (0, \mu(X))$ be given.

There exists $\varepsilon' > 0$ such that for $K \in C(X)$ with $\operatorname{diam} K < \varepsilon'$ we have $\mu(K) < t$. Otherwise, we would choose a sequence of elements of $\mu^{-1}([t, \mu(X)])$ convergent to a point, which is impossible for $t > 0$.

For this ε' take an integer k according to the uniform continuum-chainability of X.

Now, fix points A, B of $\mu^{-1}(t)$ and choose $x \in A$ and $y \in B$ arbitrarily.

There are subcontinua $A_1, \ldots, A_k$ of X each of diameter less than ε', and such that $x \in A_1$ and $y \in A_k$ and $A_i \cap A_j \neq \emptyset$ whenever $|i - j| \leqslant 1$.

We have $\mu(A_i) < t$ for $i = 1, 2, \ldots, k$. Thus, for each A_i there is a continuum $B_i \subset X$ such that $A_i \subset B_i$ and $\mu(B_i) = t$.

Let $B_0 = A$ and $B_{k+1} = B$. We have $B_i \cap B_{i+1} \neq \emptyset$, for $i = 0, \ldots, k$.

Let $q_i : [0, t] \to \mu^{-1}(t)$ be a continuous function constructed such as in Lemma 2, and such that $q_i(0) = B_i$ and $q_i(t) = B_{i+1}$, for $i = 0, \ldots, k$.

Let $q : [0, (k+1)t] \to \mu^{-1}(t)$ be such that for any $i = 0, \ldots, k$ and $u \in [it, (i+1)t]$ the equality $q(u) = q_i(u - it)$ holds.

Let us observe that q joins A with B.

To show the uniform pathwise connectedness of $\mu^{-1}(t)$, fix $\varepsilon > 0$. For this ε take the integer n according to Lemma 2. By Lemma 2, for the path q the number $(k+1)n$ is that one for which the condition $(**)$ is fulfilled.

Since the numbers n and k do not depend on the choice of points A and B of $\mu^{-1}(t)$, the proof is finished. $\square$

Now, having Theorem 1, we can answer the questions mentioned at the beginning of the paper.

Theorem 2. *The property of being a uniformly pathwise connected continuum is a Whitney property.*

Proof. Let X be a continuum with the family $\mathcal{F}$ of paths $p:[a,b]\to X$ satisfying conditions for uniform pathwise connectedness.

By Theorem 1, it suffices to show that X is uniformly continuum-chainable.

Fix $\varepsilon>0$ and take an integer n according to the uniform pathwise connectedness of X. Let $x, y\in X$ be arbitrarily chosen. There exist $p\in\mathcal{F}$ such that $p(a)=x$ and $p(b)=y$ and points $t_0=a<t_1<t_2<\cdots<t_n=b$ of $[a,b]$ such that the diameter of $A_i=p([t_{i-1},t_i])$ is $\leqslant\varepsilon$, for $i=1,2,\ldots,n$, which means that X is uniformly continuum-chainable since $x\in A_1$ and $y\in A_n$. Thus, the proof is finished. □

We say that a continuum Y is *connected by uniformly short paths* if for each positive number ε there is an integer $k=k(\varepsilon)$ such that for each pair x, y of points of Y, there is a path $p:[a,b]\to Y$ such that $p(a)=x,\ p(b)=y$, and points $t_0=a<t_1<t_2<\cdots<t_k=b$ of $[a,b]$ such that the diameter of $p([t_{i-1},t_i])$ is $\leqslant\varepsilon$, for $i=1,\ldots,k$.

This concept was introduced by Bellamy in [1], where he asked whether it was equivalent to the concept of uniform pathwise connectedness. In [3], Holmes answered the question of Bellamy constructing a continuum which is connected by uniformly short paths but not uniformly pathwise connected. Since it is easy to see that a continuum which is connected by uniformly short paths is uniformly continuum-chainable, we have the following.

Theorem 3. *The property of being a uniformly pathwise connected continuum is not a Whitney-reversible property.*

Proof. Take a continuum P which is connected by uniformly short paths but not uniformly pathwise connected. Since P is uniformly continuum-chainable then, by Theorem 1, for each Whitney map $\mu:C(P)\to[0,\infty)$ and each $t\in(0,\mu(P))$ the Whitney level $\mu^{-1}(t)$ is uniformly pathwise connected. Since the continuum P is not uniformly pathwise connected, the proof is finished. □

Question. Is it true that a continuum which is uniformly continuum-chainable is connected by uniformly short paths?

Note Added in Proof

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References

[1] D.P. Bellamy, Short paths in homogeneous continua, Topology Appl. 26 (1987) 287–291.
[2] C.L. Hagopian, L.G. Oversteegen, Continuum chainability without arcs, Houston J. Math. 21 (1995) 407–411.
[3] M.R. Holmes, There is a continuum which is connected by uniformly short paths but not uniformly path connected, Topology Appl. 42 (1991) 17–23.
[4] A. Illanes, S.B. Nadler, Hyperspaces: Fundamentals and Recent Advances, Marcel Dekker, New York, 1998.
[5] H. Kato, A note on continuous mappings and the property of J.L. Kelley, Proc. Amer. Math. Soc. 112 (1991) 1143–1148.
[6] I. Krzemińska, J.R. Prajs, A non-g-contractible uniformly path connected continuum, Topology Appl. 91 (1999) 151–158.
[7] I. Krzemińska, J.R. Prajs, On continua whose hyperspace of subcontinua is σ-locally connected, Topology Appl. 96 (1999) 53–61.
[8] W. Kuperberg, Uniformly pathwise connected continua, in: Studies in Topology, Academic Press, 1975, pp. 315–324.
[9] J. Krasinkiewicz, On the hyperspaces of snake-like and circle-like continua, Fund. Math. 83 (1974) 155–164.
[10] S.B. Nadler Jr, Hyperspaces of Sets, Marcel Dekker, 1978.
[11] S.B. Nadler Jr, Continuum Theory, Marcel Dekker, 1992.